

TWO-PHASE ANISOTROPIC FREE BOUNDARY PROBLEMS AND APPLICATIONS TO THE BELLMAN EQUATION IN 2D

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ABSTRACT. We prove Lipschitz continuity of solutions to a class of rather general two-phase anisotropic free boundary problems in 2D and we classify global solutions. As a consequence, we obtain $C^{2,1}$ regularity of solutions to the Bellman equation in 2D.

1. INTRODUCTION

One of the basic fully nonlinear 2nd order PDE is the Bellman equation, which is the equation of dynamic programming for certain optimally controlled stochastic systems. In the case of two operators the equation reads:

$$(1.1) \quad F(D^2v) := \min\{L_1v, L_2v\} = 0,$$

where L_i are constant coefficient elliptic operators

$$L_iv := \operatorname{tr}(A_i D^2v), \quad i = 1, 2 \quad \lambda I \leq A_i \leq \Lambda I.$$

The operator F is concave and solutions to (1.1) satisfy the $C^{2,\alpha}$ interior estimates of Evans and Krylov ([E, K]). In fact, the question of interior regularity for problem (1.1) in the case of two operators was first settled by Brezis and Evans in [BE].

The motivation for this note was to investigate further regularity of v and qualitative properties of the free boundary

$$\Gamma := \{x \in B_1 \mid L_1v(x) = L_2v(x)\},$$

which is a closed set in B_1 due to the Hölder continuity of D^2v .

Our main results hold in 2D and they read as follows.

Theorem 1.1. *Let v satisfy (1.1) in $B_1 \subset \mathbb{R}^2$. Assume $0 \in \Gamma$. Then one of the following alternatives holds.*

1) Γ is a smooth curve in a neighborhood of 0 and in this case v is $C^{2,1}$ near 0 and has an expansion

$$v(x) = Q + \gamma((x \cdot \nu)^+)^3 + O(|x|^{3+\alpha}),$$

with Q a third order polynomial and ν the normal to Γ at 0.

2) v is pointwise $C^{3,\alpha}$ at 0 i.e. has an expansion

$$v = P + O(|x|^{3+\alpha}),$$

with P a polynomial of degree 2 (all third derivatives vanish at 0).

L. C. is supported by NSF grant DMS-1500871. D. D. is supported by NSF grant DMS-1301535. O. S. is supported by NSF grant DMS-1200701.

The coefficients of Q , P (and γ) and the constant in $O(|x|^{3+\alpha})$ are bounded by $C\|v\|_{L^\infty(B_1)}$ with C a constant depending only on λ, Λ . As a consequence we obtain the optimal regularity of solutions to Bellman equation in 2D.

Theorem 1.2. *Assume v satisfies (1.1) and $|v| \leq 1$ in $B_1 \subset \mathbb{R}^2$. Then*

$$\|v\|_{C^{2,1}(B_{1/2})} \leq C(\lambda, \Lambda).$$

Our strategy consists in viewing this problem as a so-called two phase free boundary problem. We obtain regularity results for such type of problems that translate in our main Theorems above. These results are in fact the core of our paper and they are interesting in their own.

Precisely, assume that $u : B_1 \rightarrow \mathbb{R}$, $B_1 \subset \mathbb{R}^2$ satisfies

$$(1.2) \quad \begin{cases} L_1 u = 0 & \text{in } B_1^+(u) := \{u > 0\}, \\ L_2 u = 0 & \text{in } B_1^-(u) := \{u \leq 0\}^\circ, \\ u_\nu^+ = G(u_\nu^-, \nu) & \text{on } F(u) := \partial B_1^+(u) \cap B_1, \end{cases}$$

where the free boundary condition is understood in the viscosity sense (see Section 2 for the precise definition) and u_ν^+ and u_ν^- denote the normal derivatives of u^+ and u^- in the inward direction to $B_1^+(u)$ and $B_1^-(u)$ respectively. Here

$$L_i u(x) = \text{tr}(A_i(x) D^2 u), \quad 0 < \lambda I \leq A_i(x) \leq \Lambda I \quad \text{in } B_1$$

with A_i Hölder continuous. The function $G : \mathbb{R}^+ \times S^1 \rightarrow \mathbb{R}^+$ is continuous and it satisfies the usual ellipticity assumption

$$G(b, \nu) \text{ is strictly increasing in } b$$

and $G(b, \nu) \rightarrow \infty$ as $b \rightarrow \infty$ uniformly in ν , i.e

$$(1.3) \quad G(b, \nu) \geq \omega(b), \quad \text{with } \lim_{b \rightarrow \infty} \omega(b) = \infty.$$

Our first main result gives the Lipschitz continuity of u .

Theorem 1.3. *Let u be as above. Then*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C(\lambda, \Lambda, \omega, \|u\|_{L^\infty(B_1)}).$$

The estimate in Theorem 1.3 does not depend on the Hölder norm of the A_i and in fact our proof carries through even when the A_i are merely measurable (see Remark 2.5).

In some special cases, Theorem 1.3 is well known in any dimension. In the case when $L_1 = L_2 = \Delta$ and for a rather general class of G 's, Lipschitz continuity of a solution follows from the celebrated monotonicity formula of Alt-Caffarelli-Friedman [ACF] (see also [CJK, MP]). In a recent paper [DK], the authors prove Lipschitz continuity of the solution of a two-phase free boundary problem governed by the p -Laplacian for a special class of isotropic G 's. The case of two different operators is however more delicate and no other results on the Lipschitz regularity of solutions are available in the literature. Some partial regularity results of the free boundary are proved in [AM, F].

We remark that Theorem 1.3 cannot hold in this generality in dimension $n \geq 3$. Indeed, say for $n = 3$, it is not difficult to construct two homogeneous functions of degree less than one, that solve two different uniformly elliptic equations in complementary domains in \mathbb{R}^3 and satisfy the free boundary condition for a specific G .

After obtaining Theorem 1.3, we can also characterize the blow-up limits at free boundary points. Assume that $0 \in F(u)$ and let u^* be a blow-up limit along a subsequence $r_k \rightarrow 0$, i.e.

$$u^*(x) = \lim_{k \rightarrow \infty} \frac{u(r_k x)}{r_k}.$$

Due to Theorem 1.3 such blow-up limit functions exist and they are Lipschitz. Our second main result characterizes such blow-up limits.

Theorem 1.4. *Assume u^* is as above.*

Then either u^ is a two plane-solution*

$$(1.4) \quad u^* = a(x \cdot \nu)^+ - b(x \cdot \nu)^- \quad \text{with } a, b > 0,$$

or

$$(1.5) \quad (u^*)^- \equiv 0,$$

which means that u^ solves the one-phase problem for L_1 .*

As remarked above, Theorem 1.4 cannot hold in this generality in dimension $n \geq 3$.

If a blow-up limit u^* of u is a two-plane solution then we prove smoothness of the free boundary of u in a neighborhood of 0, provided that G is more regular. This can be achieved in any dimension by perturbation techniques by first analyzing the regularity of a transmission-type problem across $\{x_n = 0\}$ for two different linear operators (see for example [DFS1, DFS2, AM]).

Assume that G is smooth and homogenous of degree one in b . Then the alternative (1.5) above gives $u^* \equiv 0$. By compactness we obtain the following result.

Theorem 1.5. *Assume that $G(b, \nu)$ is homogenous of degree one in b , and $G(1, \nu)$ is C^1 in ν . Assume that*

$$0 \in F(u), \quad |u| \leq 1 \text{ in } B_1.$$

Then there exist $a, b \geq 0$ and a direction ν such that

$$|u(x) - [a(x \cdot \nu)^+ - b(x \cdot \nu)^-]| \leq C_0 |x|^{1+\alpha}, \quad \text{with } a = G(b, \nu).$$

The constants α, C_0 are universal, i.e. they depend only on λ, Λ , the Hölder norm of the A_i and G .

If $a \neq 0$ then $F(u)$ is $C^{1,\alpha}$ in a neighborhood of 0 and if $a = 0$ then u is pointwise $C^{1,\alpha}$ at the origin.

The strategy to prove the main theorems, Theorem 1.3 and Theorem 1.4, relies on two-dimensional topological arguments which involve intersecting the graph of the solution with a family of planes. These ideas go back to the work of Bernstein [B] and Hopf [H] and have been used more recently in [DS, S].

The paper is organized as follows. In Section 2 we prove Theorem 1.3 and Theorem 1.4. We describe our compactness method and prove Theorem 1.5 in Section 3. Section 4 is devoted to Bellman's equation and the proof of Theorem 1.1.

2. TWO-PHASE FREE BOUNDARY PROBLEMS

In this Section we provide the proofs of our main results, Theorem 1.3 and Theorem 1.4. First we introduce some standard definitions and prove some preliminary lemmas.

As in the Introduction (see (1.2)), let $u : B_1 \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy

$$(2.1) \quad \begin{cases} L_1 u = 0 & \text{in } B_1^+(u), \\ L_2 u = 0 & \text{in } B_1^-(u), \\ u_\nu^+ = G(u_\nu^-, \nu) & \text{on } F(u) \text{ when } u_\nu^- > 0. \end{cases}$$

We point out that we require the free boundary condition to hold only when $u_\nu^- \neq 0$. This is a weaker definition than it usually appears in the literature and it is understood in the following viscosity sense.

Definition 2.1. We say that u satisfies the free boundary condition

$$u_\nu^+ = G(u_\nu^-, \nu)$$

at a point $y_0 \in F(u)$ if for any unit vector ν , there exists no function $\psi \in C^2$ defined in a neighborhood of y_0 with $\psi(y_0) = 0$, $\nabla \psi(y_0) = \nu$ such that either of the following holds:

- 1) $a\psi^+ - b\psi^- \leq u$ with $a > 0$, $b > 0$ and $a > G(b, \nu)$ (i.e. u is a supersolution);
- 2) $a\psi^+ - b\psi^- \geq u$ with $a > 0$, $b > 0$ and $a < G(b, \nu)$ (i.e. u is a subsolution).

We only use comparison functions which cross the 0 level set transversally and therefore have a nontrivial negative part. For this reason the free boundary condition is preserved when taking uniform limits. It is straightforward to check that a uniform limit of solutions of (2.1) satisfies (2.1) as well.

Definition 2.2. A *two-plane solution* p to (2.1) is given by

$$p(x) = p_{x_0, \nu, a, b}(x) := a((x - x_0) \cdot \nu)^+ - b((x - x_0) \cdot \nu)^-,$$

for some $x_0 \in \mathbb{R}^2$, $\nu \in S^1$ and with

$$a = G(b, \nu), \quad \text{and} \quad a > 0, b > 0.$$

Given a function u that satisfies (2.1) and a point y away from the free boundary of u , we often consider the two-plane solution p which is tangent to the graph of u at $(y, u(y))$, i.e. such that $u(y) = p(y)$ and $\nabla u(y) = \nabla p(y)$. Notice that this two-plane solution might not always exist. It is well defined unless either $\nabla u(y) = 0$ or if $u(y) > 0$ and

$$|\nabla u(y)| \leq G(0, \nu) \quad \text{with} \quad \nu = \frac{\nabla u(y)}{|\nabla u(y)|}.$$

From the viscosity definition above and the Hopf lemma we see that a two-plane solution p cannot touch u by above (or below) on the free boundary $\{p = 0\}$ unless u and p coincide. As a consequence we obtain the following maximum principle.

Lemma 2.3 (Maximum principle). *Let Ω be a bounded domain and p a two-plane solution. If $u \leq p$ (or $u \geq p$) on $\partial\Omega$ then $u \leq p$ (respectively $u \geq p$) in Ω .*

Proof. We compare u with the continuous family of two-plane solutions $t \mapsto p(x + t\nu)$ which is strictly increasing in t . These solutions converge to $\pm\infty$ as $t \rightarrow \pm\infty$, and in view of the discussion above the maximum principle applies. \square

We assume for simplicity that $A_i(x)$ are Hölder continuous and therefore u is C^2 in $\{u > 0\} \cup \{u < 0\}$. If u is not linear in a neighborhood of a point z , then z belongs to the closure of the set $\{D^2u \neq 0\}$. In this nondegeneracy set the following topological lemma holds in 2D.

Lemma 2.4 (Connected components). *Assume $u(x_0) \neq 0$ and $D^2u(x_0) \neq 0$. Let p be a two-plane solution such that at x_0 we have $u = p$ and $\nabla u = \nabla p$. Then $\{u > p\}$ (resp. $\{u < p\}$) has two distinct connected components starting at x_0 which exit B_1 .*

Proof. The existence of the two components locally near x_0 is clear, since u is a solution to an elliptic equation and the eigenvalues have opposite sign. These components cannot be compactly included in B_1 by the maximum principle.

Moreover, if the two components reconnect further out inside B_1 then, since we are in \mathbb{R}^2 , they enclose a domain compactly included in B_1 . This domain contains a component of $\{u < p\}$ and we reach a contradiction again. \square

Remark 2.5. We remark that in the case when the A_i are merely measurable, Lemma 2.4 still holds in a dense subset of a neighborhood where the function is not linear (see Lemma 1 in [S].)

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Assume that

$$|u| \leq 1 \text{ in } B_1 \text{ and } |\nabla u(x_0)| \gg 1,$$

for some $x_0 \notin \{u = 0\}$ near 0. We will reach a contradiction.

If u is linear in a neighborhood around x_0 then, by unique continuation, u coincides with this linear function ℓ in either the set $\{\ell < 0\}$ or $\{\ell > 0\}$, and since $|\nabla \ell| \gg 1$ we contradict that $|u| \leq 1$ in B_1 . Thus u is not linear near x_0 , and without loss of generality we may assume that $D^2u(x_0) \neq 0$.

Let p be the two plane solution such that at x_0 we have $u = p$, $\nabla u = \nabla p$, and say for simplicity that

$$p = ax_2^+ - bx_2^-, \quad a = G(b, e_2),$$

with $a, b \gg 1$, which follows from our contradiction assumption and the properties of G . Then, in view of Lemma 2.4, $\{u < p\}$ has two distinct components starting at x_0 .

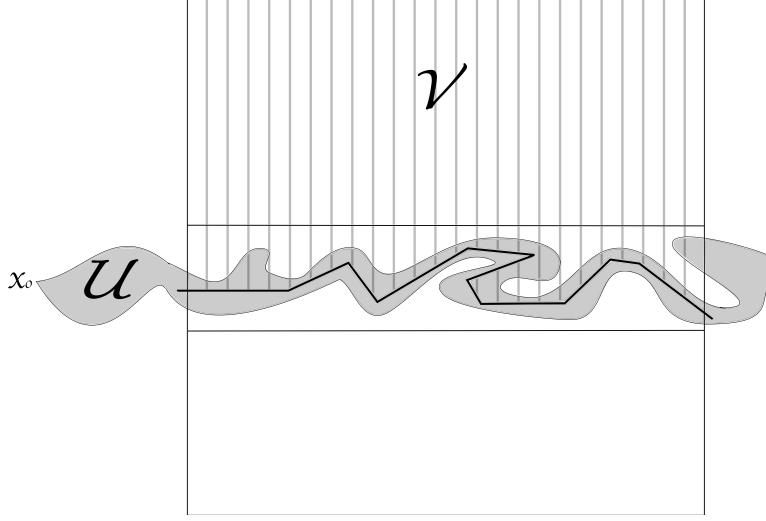
Since p grows fast in the x_2 direction and $|u| \leq 1$ it follows that one component \mathcal{U} of $\{u < p\}$ that starts at x_0 is included in the thin strip

$$\mathcal{U} \subset \left\{ -\frac{1}{b} < x_2 < \frac{1}{a} \right\}.$$

Let P be a non self-intersecting polygonal line included in \mathcal{U} that starts near x_0 and exits B_1 say on the right side of the strip above. Let R be the rectangle

$$R := \left(\frac{1}{4}, \frac{3}{4} \right) \times \left(-\frac{1}{4}, \frac{1}{4} \right),$$

and $\bar{P} \subset P \cap \bar{R}$ a part of the polygonal line P which connects the two lateral sides of R . Then \bar{P} splits R into two components and denote by \mathcal{V} the component on the top (see Figure 1).

FIGURE 1. Region \mathcal{V}

Define w in R as

$$(2.2) \quad w := \begin{cases} u & \text{in } R \setminus \mathcal{V}, \\ u & \text{in } \mathcal{V} \cap \mathcal{U}, \\ p & \text{in } \mathcal{V} \setminus \mathcal{U}. \end{cases}$$

We claim that w is a supersolution. Indeed, notice that w is a continuous function and it is a solution away from the set $\partial\mathcal{U} \cap \mathcal{V}$. On the other hand in this set $w = p$ and $w \leq p$ in a neighborhood of it, and the claim is proved.

However, w cannot be a supersolution since it stays a bounded distance away from the function ax_2^+ with $a \gg 1$, which is a strict subsolution.

Precisely we have

$$(2.3) \quad |w - ax_2^+| \leq 2.$$

Let

$$(2.4) \quad \varphi = x_2 - x_1^2 + C(\lambda, \Lambda)(x_2 - x_1^2)^2,$$

with C large such that φ is a subsolution for both L_1, L_2 . Then

$$\Psi := \omega(\gamma)\varphi^+ - \gamma\varphi^-$$

is a (classical) subsolution for the two-phase problem in a fixed neighborhood of 0. Here ω is defined in (1.3). From (2.3) we may choose γ large, universal such that a translation of the graph of Ψ is tangent by below at an interior point to the graph of w , provided that a is sufficiently large, and we reach a contradiction. \square

Remark 2.6. In the proof above we used the thin component \mathcal{U} which concentrates near the line $\{x_2 = 0\}$ in order to *glue* the solution u on one side of \mathcal{U} with the two-plane solution p on the other side of \mathcal{U} and obtain a supersolution. One can

also obtain a subsolution if in the glueing region \mathcal{U} we replace u by p i.e.

$$(2.5) \quad v := \begin{cases} u & \text{in } R \setminus (\mathcal{V} \cup \mathcal{U}), \\ p & \text{in } \mathcal{V} \cup \mathcal{U} \end{cases} \quad \text{is a subsolution.}$$

Assume now that a sequence v_m of solutions converges uniformly in ball $B_r(z)$ to a function v^* and p_m a sequence of two-plane solutions converges to p_0 . If there are connected components of $\{v_m < p_m\}$ which converge in the Hausdorff distance to a C^1 curve

$$\Sigma \subset \{v^* = p_0\} \cap B_r(z),$$

then using supersolutions and subsolutions as in (2.2)-(2.5), we see that the function which is v^* on one side of Σ and p_0 on the other side of Σ is also a solution. By the unique continuation result below we deduce that v^* coincides with p_0 .

Lemma 2.7 (Unique continuation). *Assume that u satisfies (2.1) and $u = p$ in an open set, where p is a two-plane solution. Then $u \equiv p$.*

Proof. Assume that $u = p = ax_2^+ - bx_2^-$, (with $a, b > 0$, $a = G(b, e_2)$) in a neighborhood of some point in $\{x_2 < 0\}$. Then by unique continuation $u = p$ in $\{x_2 \leq 0\}$. Since $\{u > 0\} \subset \{x_2 > 0\}$ and u is Lipschitz continuous we conclude that u^+ has an expansion at 0 (see Lemma 11.17 [CS])

$$u^+ = tx_2^+ + o(|x - y|).$$

Using the free boundary condition at 0 we find $t = a$.

Next we claim that $\{u > 0\}$ in $\{x_2 > 0\}$ near 0. Indeed, let φ be as in (2.4) and consider the comparison function

$$\Psi := (a - \epsilon)\varphi^+ - \beta\varphi^-,$$

with $\beta > 0$ small such that Ψ is a comparison subsolution in a small σ -neighborhood of 0. Notice that Ψ is strictly increasing in the e_2 direction. Using the expansion of u^+ , we see that in B_η^+ for η sufficiently small, we can compare u with translations of the rescaled subsolutions

$$\frac{\eta}{\sigma} \Psi \left(\frac{\sigma}{\eta} x \right),$$

and obtain that $u > 0$ in $B_{\eta/2}^+$, and the claim is proved.

The free boundary condition gives $u_{e_2}^+ = a$ on $\{x_2 = 0\} \cap B_{\eta/2}^+$ and by unique continuation we obtain that u coincides with p in $B_{\eta/2}^+$ and therefore in $\{x_2 > 0\}$. \square

The proof of Theorem 1.4 is a more refined version of the arguments used in Theorem 1.3. We present it below.

Proof of Theorem 1.4. We remark that u^* is a solution to our two-phase problem with constant coefficients operators, since it is the uniform limit (on compacts) of a sequence of solutions.

Denote by

$$D^+ = \nabla u^*(\{u^* > 0\}), \quad D^- = \nabla u^*(\{u^* < 0\}),$$

and D^\pm are bounded sets, since u^* is Lipschitz. Assume that $D^- \neq \emptyset$ otherwise alternative (b) holds and there is nothing to prove. We choose a direction from the origin, say e_2 for simplicity, which intersects D^- . We let

$$(2.6) \quad b := \max\{t \mid te_2 \in \overline{D^-} \text{ or } G(t, e_2)e_2 \in \overline{D^+}\}, \quad a := G(b, e_2),$$

and then $b > 0$, hence $a > 0$ as well.

Without loss of generality we assume that $a = b = 1$, since we can multiply G by a constant so that $G(1, e_2) = 1$, and let

$$p_0 := x_2.$$

The definition of a, b above says that there exists a sequence of points x_k such that the corresponding two-plane solution for u^* at x_k has normal ν_k , and slopes a_k, b_k with $\nu_k \rightarrow e_2, b_k \rightarrow 1, a_k \rightarrow 1$. Moreover there are no points for which the two-plane solution has normal e_2 and slopes strictly bigger than 1.

Theorem 1.4 will follow easily from a unique continuation argument and the open mapping theorem, once we establish the next lemma which says that the slope e_2 is in fact achieved at some point in $\{u^* > 0\} \cup \{u^* < 0\}$.

Lemma 2.8.

$$e_2 \in D^+ \cup D^-.$$

Proof. Assume by contradiction that the conclusion does not hold. Then u^* and p_0 cross transversely away from the zero level set, thus $\{u^* = p_0\}$ consists of a union of non self-intersecting C^2 curves away from $\{x_2 = 0\}$.

Notice that any connected component of $\{u^* < p_0\}$ must be unbounded.

We now divide the proof of the Lemma in three steps.

Step 1. We prove a statement about the connected components of $\{u^* < p_0\}$. We show that only a finite number of the components of $\{u^* < p_0\} \cap B_1$ intersect $B_{1/2}$, and moreover only one of these components intersects a small neighborhood of 0.

Lemma 2.9. $\{u^* < p_0\} \cap B_1$ has only one connected component near 0.

In other words, there is a ball B_δ such that all the points in $B_\delta \cap \{u^* < p_0\}$ can be joined by a path in $\{u^* < p_0\} \cap B_1$.

Proof. We distinguish two cases.

Case 1: There exists $r \in (1/2, 1)$ such that $u^*(\pm r, 0) \neq 0$. Let

$$\mathcal{R} := [-r, r] \times [-\delta, \delta]$$

be a rectangle with δ small such that on each lateral side of \mathcal{R} we have either $\{u^* > p_0\}$ or $\{u^* < p_0\}$. Denote by T^+ and T^- the top and bottom sides of \mathcal{R} ,

$$T^+ = [-r, r] \times \{\delta\}, \quad T^- = [-r, r] \times \{-\delta\}.$$

We remark that, since u^* and p_0 cross transversally away from the line $x_2 = 0$, there are only a finite number of components of $\{u^* < p_0\} \cap B_1$ which intersect $\partial\mathcal{R}$.

Claim: Each connected component \mathcal{U} of $\{u^* < p_0\} \cap \mathcal{R}$ must intersect T^+ .

If \mathcal{U} intersects a lateral side of \mathcal{R} then it intersects T^+ as well. Assume that \mathcal{U} intersects only the bottom boundary T^- of \mathcal{R} (as in the picture below, Figure 2).

Consider the family of linear functions

$$\ell_s := (1 + s)x_2 - 2s, \quad \text{for } s > 0,$$

and notice that in B_1 we have $p_0 > \ell_s$ for all $s > 0$ and $u^* > \ell_s$ for s large. We decrease s till the graph of ℓ_s touches the graph of u^* by below at a point $(x^*, u^*(x^*))$.

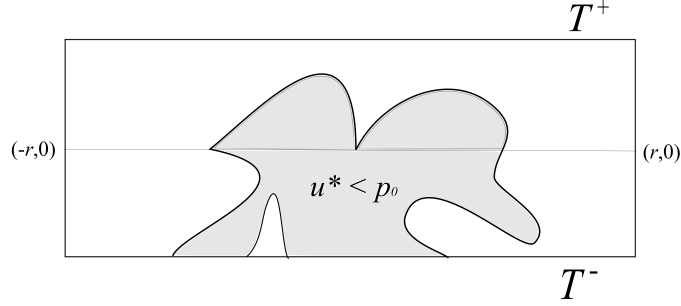


FIGURE 2.

If $x^* \in T^-$ then we find $\nabla u^*(x^*) = te_2$ for some $t \geq 1 + s > 1$, and we contradict that $b = G(1, e_2) = 1$, with b defined in (2.6).

Similarly, if $x^* \in \{x_2 > -\delta\}$ and $u^*(x^*) \neq 0$ then $\nabla u^*(x^*) = (1 + s)e_2$, again a contradiction.

If $u^*(x^*) = 0$ then $(u^*)^+$ has a linear expansion near x^*

$$(u^*)^+(x) = t(x_2 - x_2^*)^+ + o(|x - x^*|), \quad \text{for some } t \geq 1 + s.$$

This implies $te_2 \in \overline{D^+}$ and we reach a contradiction as above, and the claim is proved.

Analogously, we obtain that each component of $\{u^* > p_0\} \cap \mathcal{R}$ must intersect T^- .

Now let us assume by contradiction that two distinct components \mathcal{U}_1 and \mathcal{U}_2 of $\{u^* < p_0\} \cap \mathcal{R}$ intersect any neighborhood of the origin i.e.

$$(2.7) \quad 0 \in \partial \mathcal{U}_1 \cap \partial \mathcal{U}_2.$$

From the claim we find points $y_i \in \mathcal{U}_i \cap T^+$. Let z be a point on the segment $[y_1, y_2]$ which belongs to the set $\{u^* = p_0\}$. Since u^* and p_0 cross transversally and $\nabla u^*(z) \neq te_2$ with $t \geq 1$ we see that $z \in \partial \mathcal{V}$ for some component \mathcal{V} of $\{u^* > p_0\} \cap \mathcal{R}$. Using the claim again, we can find a non self-intersecting polygonal line which connects z with the bottom T^- and which is included in \mathcal{V} except at the initial point z . This polygonal line splits the rectangle \mathcal{R} into two separate regions one containing \mathcal{U}_1 and the other \mathcal{U}_2 and we contradict (2.7).

Case 2: $u^* = 0$ on a segment $\ell \subset \{x_2 = 0\}$.

We show that there exists a point $x_0 \in \ell$ and $\delta > 0$ small such that

$$(2.8) \quad u^* < p_0 \quad \text{in } B_\delta^+(x_0), \quad u^* > p_0 \quad \text{in } B_\delta^-(x_0).$$

Once we establish this, the arguments from *Case 1* carry through as before. Indeed if (2.8) holds, we can construct a rectangle \mathcal{R} for which the claim holds since we can guarantee that if \mathcal{U} intersects a lateral side of \mathcal{R} then it intersects T^+ as well.

We show only the first inequality in (2.8) since the second follows in the same way. The function $(u^*)^+$ has an expansion in $\{x_2 > 0\}$ near a point $y \in \ell$ as

$$(u^*)^+ = tx_2^+ + o(|x - y|).$$

If $t < 1$ then we can compare u^* with its *harmonic replacement* in $B_\eta^+(y)$ for some small η and we easily obtain that $(u^*)^+ \leq \frac{t+1}{2}x_2 < (1 - \epsilon)p_0$ in $B_{\eta/2}^+(y)$.

If $t \geq 1$ then as in the proof of Lemma 2.7 and find that $u^* > 0$ in $B_\eta^+(y)$.

As before, if $(u^*)_\nu^+ < 1$ at any point on $\ell \cap B_\eta(y)$, then we obtain the desired conclusion in a neighborhood of that point. If $(u^*)_\nu^+ > 1$ at any point on the same segment then we find $\tilde{t}e_2 \in \overline{D^+}$ for some $\tilde{t} > 1$ and we reach a contradiction. Otherwise, $(u^*)_\nu^+ = 1$ on the whole segment, and by unique continuation we obtain that u^* coincides with x_2 in $B_{\eta/4}^+(y)$, thus $e_2 \in D^+$, contradiction.

In conclusion (2.8) holds and Lemma 2.9 is proved. \square

Step 2. From the definitions of u^* , D^\pm we easily obtain.

Lemma 2.10. *There exists a sequence of points $y_m \rightarrow 0$ and blow-up functions u_m such that $D^2 u_m(y_m) \neq 0$ and $\nabla u_m(y_m) \rightarrow e_2$. This means that the two-plane solutions p_m corresponding to y_m satisfy $p_m \rightarrow p_0$.*

Let x_k be a point such that $u^*(x_k) \neq 0$, $\nabla u^*(x_k)$ is close to e_2 and $D^2 u^*(x_k) \neq 0$. Since

$$u_m(x) := \frac{1}{r_m} u(r_m x) \quad r_m \rightarrow 0,$$

converge uniformly (in C^2) to u^* in a neighborhood of x_k , we find that $y_m := r_m x_k$ and u_m for some large m depending on k has the required properties of the lemma.

Step 3. We apply the two components lemma 2.4 for u_m at y_m and recall that $u_m \rightarrow u^*$, $p_m \rightarrow p_0$, $y_m \rightarrow 0$. In view of Lemma 2.9 one of the two connected components of $\{u_m < p_m\}$ must concentrate as $m \rightarrow \infty$ near the set $\{u^* = p_0\}$ in a small ball B_δ around the origin. Next we want to apply Remark 2.6 and reach a contradiction. As in Lemma 2.9 we consider the two cases.

Case 1: There exists $r \in (0, \delta)$ such that $u^*(\pm r, 0) \neq 0$. Then we can find a point $z \in \partial B_r \cap \{u^* = p_0\}$ and a neighborhood of z away from $\{x_2 = 0\}$ where we can apply Remark 2.6.

Case 2: $\{u^* = p_0\}$ contains a segment on $x_2 = 0$ and this segment is the limit of a sequence of connected components of $\{u_m < p_m\}$. Then Remark 2.6 applies again at some interior point z of this segment. \square

End of proof of Theorem 1.4. In conclusion $e_2 \in D^+ \cup D^-$, and there exists x_0 such that $\nabla u^*(x_0) = e_2$. Since we are in 2D, in view of the open mapping Theorem and the definition (2.6) of $b = 1$, we conclude that $\nabla u^* \equiv e_2$ in a neighborhood of x_0 . Hence u^* is linear in such neighborhood and by Lemma 2.7 we conclude that $u^* \equiv p_0$. \square

3. COMPACTNESS ARGUMENTS

The purpose of this section is to obtain Theorem 1.5. The key ingredient is Proposition 3.2 below. First, we observe that by the method of [DFS1, DFS2] (which applies also for two different operators) we have the following perturbation result which holds in any dimension.

Proposition 3.1. *Assume that $p = p_{0,\nu,a,b}$ is a two-plane solution and G is C^1 in a neighborhood of (b, ν) . There exists ϵ_0 small depending only on G such that if u is a solution and*

$$|u - p| \leq \epsilon \quad \text{in } B_1,$$

for some $\epsilon \leq \epsilon_0$, then $F(u)$ is a $C^{1,\beta}$ graph in $B_{1/2}$ with $C^{1,\beta}$ norm bounded by $C\epsilon$.

Indeed, the linearization becomes a transmission problem for different constant coefficients operators across the plane $x \cdot \nu = 0$ with the jump condition

$$(3.1) \quad G(b, \nu) v_\nu^+ = v_\nu^- G_1(b, \nu) b + v_\tau \cdot G_\nu(b, \nu).$$

This is obtained formally by expanding in ϵ the free boundary condition for the perturbed solution (say $0 \in F(u), \nu(0) = e_n$),

$$u = a(x_n + \epsilon v)^+ - b(x_n + \epsilon v)^-, \quad a = G(b, e_n).$$

This leads to the jump condition (3.1).

This linear problem is invariant under translations along directions perpendicular to ν , and therefore solutions are smooth in these directions. A viscosity approach to this type of transmission problem has been developed in [DFS2] (see Theorem 3.2 and the main Proposition 3.5.) In that context G did not depend on ν . However, all arguments are easily adapted to the case when G depends also on ν (see also [AM].)

Next we assume that G is homogenous of degree 1 in the b variable. In view of Theorem 1.4 and Proposition 3.1 we obtain

Proposition 3.2. *Assume that u is a solution in $B_1 \subset \mathbb{R}^2$ and $u(0) = 0$, $|u| \leq 1$. Then there exists a two-plane solution p (here we include also $p \equiv 0$) such that*

$$|u - p| \leq C_0 |x|^{1+\alpha}$$

for some C_0 depending only on G .

Proof. We know that u is Lipschitz in $B_{1/2}$, i.e. $|\nabla u| \leq C$, and we pick ϵ_0 sufficiently small such that Proposition 3.1 applies with this ϵ_0 for all planes p with either a or b in $[\frac{1}{4}, C]$.

Claim: There exists $\rho \in (0, \frac{1}{2})$ depending only on G such that in some ball B_{r_0} with $r_0 \in (\rho, \frac{1}{2})$ depending also on u we have

$$\text{either } |u - p| \leq \epsilon_0 r_0 \quad \text{or} \quad |u| \leq \frac{1}{2} r_0,$$

for some two-plane solution $p_{a,b}$ with either a or b in $[\frac{1}{4}, C]$.

Indeed if this property does not hold for a sequence of $\rho_k \rightarrow 0$ and functions u_k , then we can extract a subsequence which converges uniformly to a limiting solution u_∞ . By Theorem 1.4 we have

$$u_\infty = p_{a,b} + o(|x|).$$

If either a or b are greater than $\frac{1}{4}$ then we contradict the first alternative for some k large, and if both a, b are less than $1/4$ then we contradict the second alternative, and the claim is proved.

We choose $\alpha > 0$ small such that $1/2 \leq \rho^\alpha$ and $\alpha \leq \beta$ with β given by Proposition 3.1. Notice that if the first alternative holds then the conclusion of Proposition 3.2 is clearly satisfied for some large C_0 . If the second alternative holds then the rescaling

$$\tilde{u} := r^{-(1+\alpha)} u(rx), \quad r = r_0,$$

is a solution in B_1 and $|\tilde{u}| \leq 1$. Now we apply the claim to \tilde{u} and repeat this process either

1) a finite number of times with $r = r_k$, $r_{k+1}/r_k \in (\rho, \frac{1}{2})$ and stop the first time we end up with the first alternative for \tilde{u} . Then

$$|u - p_{\nu,a,b}| \leq C_0|x|^{1+\alpha} \quad \text{in } B_r, \quad |a|, |b| \leq Cr^\alpha,$$

and $|u|, |p|$ are bounded by $C|x|^{1+\alpha}$ outside B_r and the conclusion follows.

2) an infinite number of times and \tilde{u} satisfies the second alternative indefinitely. In this case we clearly satisfy Proposition 3.2 with $p \equiv 0$. \square

Remark 3.3. Proposition 3.2 implies that for any point $x_0 \in B_{1/4}$ we have

$$(3.2) \quad |u - p_{x_0}| \leq C|x - x_0|^{1+\alpha}$$

where p_{x_0} is the two-plane solution at x_0 , and C is a universal constant.

Indeed, let r be the distance from x_0 to $\{u = 0\}$ and z a point where the distance is realized. Denote by p_z the two plane solution at z given by Proposition 3.2, hence $|u - p_z| \leq Cr^{1+\alpha}$ in $B_r(x_0)$. This gives $|p_{x_0} - p_z| \leq Cr^{1+\alpha}$ in $B_r(x_0)$ which implies

$$|p_{x_0} - p_z| \leq Cr^\alpha|x - x_0| \quad \text{outside } B_r(x_0),$$

and the claim easily follows.

4. REDUCTION TO A TWO-PHASE FREE BOUNDARY PROBLEM

In this section we show that problem (1.1) can be reduced to a two-phase problem of the form (1.2), for a specific G and we finally obtain our main Theorem 1.1.

Without loss of generality we may assume that

$$L_1v = \Delta v \quad \text{and} \quad L_2v = v_{11} + m v_{22}, \quad \text{for some } m \geq 1.$$

We establish the following result.

Proposition 4.1. *The function*

$$u = v_{22}$$

satisfies the two-phase free boundary problem

$$\begin{aligned} L_1u &= 0 \quad \text{in } \{u > 0\}, & L_2u &= 0 \quad \text{in } \{u < 0\}, \\ u_\nu^+ &= (1 + (m-1)\nu_2^2)u_\nu^- \quad \text{on } \Gamma. \end{aligned}$$

The free boundary condition can be easily deduced when the free boundary $\Gamma \in C^1$. Indeed let $w_1 := L_2v = (m-1)u^+$, $w_2 := L_1v = (m-1)u^-$. If ν denotes the normal to Γ then

$$L_1L_2v = L_iw_i = (w_i)_\nu^+ A_i(\nu, \nu) d\mathcal{H}^1|_\Gamma.$$

This computation holds in any dimension and gives the free boundary condition

$$(w_1)_\nu^+ A_1(\nu, \nu) = (w_2)_\nu^+ A_2(\nu, \nu) \quad \text{on } \Gamma.$$

In fact the proof below of Proposition 4.1 applies in any dimension.

We prove Proposition 4.1 by approximating F by C^2 operators

$$F_\epsilon(D^2u) = u_{11} + h_\epsilon(u_{22})$$

with $h_\epsilon(s)$ a smoothing of the Lipschitz function

$$h_0(s) = s^+ - m s^-.$$

Proof. We differentiate twice in the e_2 direction and obtain that a solution to $F_\epsilon(D^2v) = 0$ satisfies

$$\Delta v_{22} + h'_\epsilon(v_{22})v_{2222} + h''_\epsilon(v_{22})v_{222}^2 = 0,$$

hence $u = v_{22}$ satisfies

$$(4.1) \quad \Delta u + h'_\epsilon(u)u_{22} + h''_\epsilon(u)u_2^2 = 0.$$

By $C^{2,\alpha}$ estimates, the solutions u above are uniformly Hölder, hence converge uniformly to \bar{u} as $\epsilon \rightarrow 0$. In order to find the equation for \bar{u} we need to find a family of solutions of (4.1) (or subsolutions and supersolutions) that converge with ϵ .

First we look for one-dimensional functions $g_\epsilon(x \cdot \nu)$ which solve (4.1). For simplicity of notation we drop the subindex from g , h and we obtain

$$[1 + \nu_2^2 h'(g)] g'' + \nu_2^2 h''(g) g'^2 = 0,$$

or

$$([1 + \nu_2^2 h'(g)]g')' = 0,$$

hence

$$\frac{g'(t_2)}{g'(t_1)} = \frac{1 + \nu_2^2 h'(g(t_1))}{1 + \nu_2^2 h'(g(t_2))}$$

which means that the derivative of g jumps by a factor of $\nu_1^2 + m\nu_2^2$ after passing through 0. Thus any two-plane function

$$a(x \cdot \nu)^+ - b(x \cdot \nu)^-, \quad \text{with } a = (\nu_1^2 + m\nu_2^2)b \geq 0,$$

is the uniform limit of solutions g_ϵ and therefore it is a comparison function for \bar{v} .

Now we can slightly modify the comparison function above and obtain a perturbed family for which the free boundary is a large sphere instead of a hyperplane. Consider functions $g_\epsilon(d)$ where d is the signed distance to a sphere of radius δ^{-3} passing through the origin and with inner normal ν , with $d > 0$ inside the sphere. We let $g(t)$ such that it satisfies the ODE

$$[1 + (\nu_2^2 + \delta)h'(g)]g' = 1 + \delta t, \quad g(0) = 0.$$

Using that h is concave and $h(0) = 0$ for $t > 0$ we obtain that $g'' > c\delta$, $g' < C$ and one can easily see as in the computation above that $g(d)$ is a subsolution in B_1 .

As $\epsilon \rightarrow 0$, g_ϵ converges uniformly to

$$\bar{g}(t) := \left(t + \frac{\delta}{2}t^2\right)^+ - b\left(t + \frac{\delta}{2}t^2\right)^- \quad \text{with } b = \frac{1}{\nu_1^2 + m\nu_2^2} + O(\delta),$$

and \bar{g} is a comparison subsolution for \bar{v} . We obtain the desired conclusion since any comparison subsolution $a\varphi^+ - b\varphi^-$ with $\nabla\varphi(0) = \nu$ and $a > G(b, \nu)$ appearing in the viscosity definition of (2.1) can be touched strictly by below at the origin by a multiple and a rescaling of $\bar{g}(d)$ (with δ sufficiently small). \square

The proof of our main Theorem 1.1 now follows immediately.

Proof of Theorem 1.1 From Proposition 3.1 and Remark 3.2 we know that $u := v_{22}$ satisfies (3.2). Then

$$L_2v = (m-1)u^+ =: w$$

and for each x_0 in $B_{1/4}$ there exists a point $z \in \{u = 0\}$ such that

$$|w - \sigma((x - z) \cdot \nu)^+| \leq C|x - x_0|^{1+\alpha}, \quad \text{for some } \sigma \geq 0.$$

Let $q = \gamma[(x - z) \cdot \nu]^3$ be such that $L_2 q = \sigma((x - z) \cdot \nu)^+$ and then the right hand side of $L_2(v - q)$ is pointwise $C^{1,\alpha}$ at x_0 . By the pointwise Schauder estimates for linear equations we obtain that $v - q$ is pointwise $C^{3+\alpha}$ at x_0 . \square

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